ON THE MOTION OF A RIGID BODY UNDER THE ACTION OF ROTATION OF AN INTERNAL FLYWHEEL

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PMM Vol.28, № 1, 1964, pp.171-174

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(Received October 16, 1963)

We consider below the solution of the problem of the finite precession of an *unsymmetric* rigid body under the action of the rotation of an internal flywheel whose axis is arbitrarily fixed in the body. It is shown that in the general case of rotation of a flywheel around an arbitrary axis, the supporting body accomplishes a pure rotation around another axis called the accompanying axis. Formulas are obtained for the determination of the direction of the flywheel axis in the body from the given magnitude of the spatial precession of the supporting body.

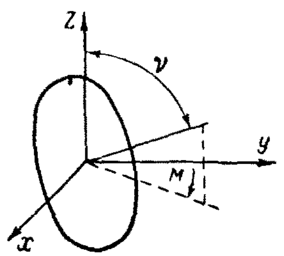

Fig. 1

Let us assume that the direction of flywheel rotation axis relative to the principal inertia axes x, y, z of the body (see Fig. 1), is determined by the angles μ and ν, that the angular velocity of the flywheel relative to the frame is $\Omega(t)$, and that its moment of inertia relative to the axis of rotation is J. Moreover, we shall consider that initially the supporting body and the flywheel were stationary, i.e. the sum of their kinetic moments equalled zero. Then, from the law of conservation of kinetic moments, for the supporting body-flywheel system we have

$$A\omega_x + J\Omega(t)\sin\nu\sin\mu = 0 \qquad B\omega_y + J\Omega(t)\sin\nu\cos\mu = 0 \qquad C\omega_z + J\Omega(t)\cos\nu = 0 \tag{1}$$

where A, B and C are the principal moments of inertia of the supporting body together with the flywheel, and ω_x, ω_y and ω_z are the projections of the angular velocity vector of the body onto its principal axes x, y and z. In order to find the motion of the body under the action of rotation of the flywheel, we must replace the projections of the angular velocity in Equations (1) by their expressions in terms of the angles and product of the angles determining the orientation of the supporting body in space. In the problem considered it is convenient to define the orientation of the body by means of the Cayley-Klein parameters α and β, connected with the Euler angles θ, ψ and φ by the relations

$$\alpha = \cos{}^1/_2\theta e^{{}^1/_2 i(\psi+\varphi)} \qquad \beta = i\sin{}^1/_2\theta e^{{}^1/_2 i(\psi-\varphi)} \tag{2}$$

The equation describing the motion of the body can be written in the form [1]

$$\alpha^{\cdot} = {}^1/_2 i\omega_z\alpha + {}^1/_2(\omega_y + i\omega_x)\beta, \qquad \beta^{\cdot} = -{}^1/_2 i\omega_z\beta - {}^1/_2(\omega_y - i\omega_x)\alpha \tag{3}$$

By substituting here the values of ω_x, ω_y and ω_z by their expressions from Equations (1), we obtain

$$\begin{aligned} \alpha^{\cdot} &= -i\frac{J\Omega\cos\nu}{2C}\alpha - \frac{J\Omega\sin\nu}{2}\left(\frac{\cos\mu}{B} + i\frac{\sin\mu}{A}\right)\beta \\ \beta^{\cdot} &= i\frac{J\Omega\cos\nu}{2C}\beta + \frac{J\Omega\sin\nu}{2}\left(\frac{\cos\mu}{B} - i\frac{\sin\mu}{A}\right)\alpha \end{aligned} \tag{4}$$

From the form of this system it follows that the precession of the frame of the supporting body is completely determined from the angle of precession of the flywheel and does not depend on the nature of the velocity variations of its rotation. Formally, this follows from the fact that in the described system we can eliminate the time t and as the independent variable introduce the precession angle of the flywheel

$$\tau = \int_0^t \Omega(t)\,dt \tag{5}$$

After this, system (4) takes the form

$$\begin{aligned} \alpha' &= -i\frac{J\cos\nu}{2C}\alpha - \frac{J\sin\nu}{2}\left(\frac{\cos\mu}{B} + i\frac{\sin\mu}{A}\right)\beta \\ \beta' &= i\frac{J\cos\nu}{2C}\beta + \frac{J\sin\nu}{2}\left(\frac{\cos\mu}{B} - i\frac{\sin\mu}{A}\right)\alpha \end{aligned} \tag{6}$$

Here the prime denotes differentiation with respect to τ. Equations (6) represent a homogeneous, linear system of equations with constant coefficients.

As usual, we shall seek its solution in the form

$$\alpha = Me^{\gamma\tau}, \qquad \beta = Ne^{\gamma\tau} \tag{7}$$

By setting up the characteristic equation of the system (6), we find

$$\gamma = \pm\frac{i}{2}\left[\left(\frac{J\cos\nu}{C}\right)^2 + \left(\frac{J\sin\nu\cos\mu}{B}\right)^2 + \left(\frac{J\sin\nu\sin\mu}{A}\right)^2\right]^{1/2} = \pm\frac{i\lambda}{2} \tag{8}$$

It is easy (referring to the original system (1)) to note that the expression under the radical is nothing but the relation ω^2/Ω^2, where $\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2$, so that from (8) there follows a relation which is valid at any instant of time

$$\omega / \Omega = \lambda = \text{const} \tag{9}$$

The solution for the parameters α and β can be written as

$$\alpha = M_1 e^{1/2 i\lambda\tau} + M_2 e^{-1/2 i\lambda\tau}, \qquad \beta = N_1 e^{1/2 i\lambda\tau} + N_2 e^{-1/2 i\lambda\tau} \tag{10}$$

The constants of integration M and N are connected by the following dependencies

$$N_1 = -ik_1M_1, \qquad N_2 = -ik_2M_2 \tag{11}$$

Here, the coefficients k_1 and k_2 are defined by Formulas

$$k_{1,2} = \left(\frac{\cot\nu}{C} \pm R\right)\left(\frac{\cos\mu}{B} + i\frac{\sin\mu}{A}\right)^{-1} \qquad \left(R = \left(\frac{\cot^2\nu}{C^2} + \frac{\cos^2\mu}{B^2} + \frac{\sin^2\mu}{A^2}\right)^{1/2}\right) \tag{12}$$

However, we should keep in mind condition

$$\bar{\alpha}\alpha + \bar{\beta}\beta = 1 \tag{13}$$

from which, after substituting in it solutions (10) and taking (11) into account, it follows

$$M_1\overline{M_1}\,(1 + k_1\bar{k}_1) + M_2\overline{M_2}\,(1 + k_2\bar{k}_2) = 1 \tag{14}$$

Thus, the Cayley-Klein parameters α and β, expressed as functions of the precession angle τ of the controlling flywheel, have the form

$$\alpha = M_1e^{1/_2 i\lambda\tau} + M_2e^{-1/_2 i\lambda\tau}, \qquad \beta = -\, ik_1M_1e^{1/_2 i\lambda\tau} - ik_2M_2e^{-1/_2 i\lambda\tau} \tag{15}$$

Since the moduli of the constants M_1 and M_2 satisfy Equation (14), the obtained solutions actually contain only three unknown real constants of integration, corresponding to the three degrees of freedom of the spatial precession of the body. Let us multiply the first of Equations (15) in turn by k_1 and k_2 and each time subtract from it the second equation. After this we get

$$(k_1 - k_2)\, M_2e^{-1/_2 i\lambda\tau} = k_1\alpha - i\beta, \qquad (k_2 - k_1)\, M_1e^{1/_2 i\lambda\tau} = k_2\alpha - i\beta \tag{16}$$

Hence there immediately ensue the two complex integrals

$$(k_1\alpha - i\beta)\, e^{1/_2 i\lambda\tau} = \text{const}, \qquad (k_2\alpha - i\beta)\, e^{-1/_2 i\lambda\tau} = \text{const} \tag{17}$$

In order to solve the problem in Euler angles we should express α and β in accordance with relations (2), then the integrals (17) take the form

$$\begin{aligned} &[k_1 \cos {}^1/_2\theta e^{1/_2 i(\psi+\varphi)} + \sin {}^1/_2\theta e^{1/_2 i(\psi-\varphi)}]\, e^{1/_2 i\lambda\tau} = \text{const} \\ &[k_2 \cos {}^1/_2\theta e^{1/_2 i(\psi+\varphi)} + \sin {}^1/_2\theta e^{1/_2 i(\psi-\varphi)}]\, e^{-1/_2 i\lambda\tau} = \text{const} \end{aligned} \tag{18}$$

If we multiply the last two integrals by each other and replace k_1 and k_2 by their representations, we get an integral not containing the argument τ

$$\begin{aligned} &[C^{-1} \cot \nu \sin\theta - \cos\theta\, (B^{-1} \cos\varphi \cos\mu + A^{-1} \sin\varphi \sin\mu) - \\ &- i\,(B^{-1} \sin\varphi \cos\mu - A^{-1} \cos\varphi \sin\mu)]\, e^{i\psi} = \text{const} \end{aligned} \tag{19}$$

This integral contains all the three Euler angles θ, ψ and φ. However, we can obtain an integral, which also is an integral of the problem, and which contains only the two Euler angles θ and φ by expressing, for example, the modulus of any of the integrals in (18)

$$\cos\theta + \tan\nu \sin\theta\, (CB^{-1} \cos\varphi \cos\mu + CA^{-1} \sin\varphi \sin\mu) = \text{const} \tag{20}$$

In order to realize the precession of the body from some initial position θ_0, ψ_0, φ_0 to a required final position θ_1, ψ_1, φ_1 it is necessary to choose in an appropriate way the flywheel axis setting angles μ and ν and also its precession angle τ. Here we should keep in mind that both the values of the moments of inertia A, B and C, as well as the directions of the principal axes in the body, generally speaking, depend on the flywheel setting angles μ and ν. However, for comparatively small-sized flywheels this dependence can be neglected in practice. Then, in order to determine the angles μ and ν, we can directly use the obtained integrals (19) and (20). The final formulas for determining the setting angles of the flywheel axis can be written as

$$\begin{aligned} B \tan\mu = A\ &[\sin \Delta\psi\, (\cos\theta_1 \cos\varphi_1 \sin \theta_0 - \cos\theta_0 \cos\varphi_0 \sin \theta_1) + \\ &+ \cos \Delta\psi\, (\sin\varphi_1 \sin\theta_0 + \sin\varphi_0 \sin\theta_1) - \sin\theta_1 \sin\varphi_1 - \sin\theta_0 \sin\varphi_0] \times \\ &\times [\sin \Delta\psi\, (-\cos\theta_1 \sin\varphi_1 \sin\theta_0 + \cos\theta_0 \sin\varphi_0 \sin\theta_1) + \\ &+ \cos \Delta\psi\, (\cos\varphi_1 \sin\theta_0 + \cos\varphi_0 \sin\theta_1) - \sin \theta_1 \cos\varphi_1 - \sin\theta_0 \cos\varphi_0]^{-1} \end{aligned} \tag{21}$$

$$C \tan \nu = [\cos\theta_0 - \cos\theta_1]\,[B^{-1}\cos\mu\,(\sin\theta_1\cos\varphi_1 - \sin\theta_0\cos\varphi_0) + A^{-1}\sin\mu\,(\sin\theta_1\sin\varphi_1 - \sin\theta_0\sin\varphi_0)]^{-1} \tag{22}$$

$$\Delta\psi = \psi_1 - \psi_0$$

By knowing μ and ν it is easy to compute λ and $k_{1,2}$, and then, using any of the integrals in (18), to find also the angle τ. Thus, the derived formulas make it possible to determine the flywheel setting angles and its precession angle τ, needed for realizing the required reorientation of the supporting body.

In order to elucidate the general nature of the motion of the supporting body under the action of an internal flywheel rotation, we return again to integrals (17). Expression (12) is written as

$$k_{1,2} = \left(1 \pm \frac{i\cot\nu}{CR}\right)\left(\frac{\cos^2\mu}{B^2} + \frac{\sin^2\mu}{A^2}\right)^{-1/2} R\exp\left[-i\tan^{-1}\left(\frac{B}{A}\tan\mu\right)\right] \tag{23}$$

Let us introduce the new angles μ^* and ν^* defined by Formulas

$$\tan\mu^* = \frac{B}{A}\tan\mu, \qquad \cos\nu^* = \frac{\cot\nu}{CR}, \qquad \sin\nu^* = \frac{1}{R}\left(\frac{\cos^2\mu}{B^2} + \frac{\sin^2\mu}{A^2}\right)^{1/2} \tag{24}$$

Then, for the coefficients k_1 and k_2 we get

$$k_1 = \cot {}^1/_2\nu^* e^{-i\mu^*}, \qquad k_2 = -\tan {}^1/_2\nu^* e^{-i\mu^*} \tag{25}$$

By substituting these expressions into integrals (17) and by multiplying them by some constant factors, we write them in the form

$$\begin{aligned} (\alpha\cos {}^1/_2\nu^* e^{-1/2 i\mu^*} - i\beta\sin {}^1/_2\nu^* e^{1/2 i\mu^*})\, e^{1/2 i\lambda\tau} &= \text{const} \\ (-\alpha i\sin {}^1/_2\nu^* e^{-1/2 i\mu^*} + \beta\cos {}^1/_2\nu^* e^{1/2 i\mu^*})\, e^{-1/2 i\lambda\tau} &= \text{const} \end{aligned} \tag{26}$$

The expressions in the brackets are the Cayley-Klein parameters of the total precession from a system of stationary axes to a system of axes connected to the supporting body but not coinciding with its principal inertia axes. We can call this latter system of axes connected to the body, the accompanying system since one of its axes, by forming the angles μ^* and ν^* with the principal inertia axes of the body, accompanies the flywheel axes in the body; $\tan\mu^*$ and $\tan\nu^*$ are determined from the relations (24).

The angle μ^*, although it will not be an Euler angle (a precession angle), actually differs from it by a constant magnitude. Therefore, by denoting the Euler angles in the accompanying system of axes relative to the stationary system by $\theta^*, \psi^*, \varphi^*$, and the corresponding Cayley-Klein parameters by α^* and β^*, we can write the integrals (26) in the form

$$\alpha^* e^{1/2 i\lambda\tau} = \text{const}, \qquad \beta^* e^{-1/2 i\lambda\tau} = \text{const} \tag{27}$$

or, in the angles θ^*, ψ^* and φ^*

$$\cos {}^1/_2\theta^* e^{1/2 i(\psi^* + \varphi^* + \lambda\tau)} = \text{const}, \qquad \sin {}^1/_2\theta^* e^{1/2 i(\psi^* - \varphi^* - \lambda\tau)} = \text{const} \tag{28}$$

Hence, it becomes clear at once that the angles θ^* and ψ^* remain constant during the motion, while the angle φ^* changes linearly with τ, i.e.

$$\theta^* = \theta_0^* = \text{const}, \qquad \psi^* = \psi_0^* = \text{const}, \qquad \varphi^* = \varphi_0^* - \lambda\tau \tag{29}$$

Thus, under the rotation of an internal flywheel around an arbitrary fixed axis, the frame of the supporting body accomplishes a pure rotation around another fixed axis whose direction in space is given by the angles θ^* and ψ^*, and in the body by the angles μ^* and ν^*. We note that the integrals (19) and (20) obtained earlier express nothing but the constancy of the cosines of the angles between the axis of rotation of the body and the stationary coordinate axes. It is obvious that the rotation axis of the body coincides with the flywheel axis only in the case either when $A = B = C$ or when the flywheel axis is directed along one of the principal inertia axes of the body.

By the Euler theorem [2] every reorientation of the rigid body can be kinematically realized by means of its precession around some stationary axis. The results obtained indicate that similar precessions can be effected dynamically with the help of one flywheel with a fixed axis. In order to find the required setting angles of the flywheel axis in the body from the given initial and final positions of the supporting body, on the basis of kinematic constructions [2], we should determine the rotation axis of the supporting body, i.e. find the angles μ^* and ν^*, and then, by using formulas (24), compute the desired angles μ and ν.

Thus, calculation of precession of the supporting body around an arbitrary axis under the action of the rotation of an internal flywheel, can be carried out by relations analogous to the calculation of the precession of the body around one of its principal inertia axes, i.e. using the equations of planar precession when calculating spatial precession.

In this case the last of Equations (29) will have the form

$$J\dot{\tau} + \varphi^* (C^2 \cos^2 \nu^* + B^2 \sin^2 \nu^* \cos^2 \mu^* + A^2 \sin^2 \nu^* \sin^2 \mu^*)^{1/2} = \text{const} \qquad (30)$$

where the square root plays the role of some reduced moment of inertia of the supporting body relative to its rotation axis. Of course, this conclusion remains valid only in the case when the initial kinetic moment of the supporting body-wheel system equals zero.

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Translated by N.H.C.